A NEW ASPECT IN THE CALCULATION OF TOOTHED GEARING

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Summary:

This paper deals with the calculation of the numbers of teeth of the wheels in a gearing with a prescribed gear ratio, equal to the quotient of two integers k/l. The known methods of decomposing k and l into prime factors, and of developing k/l in continued fractions often fail to give a practical and exact solution. A new method is described, based on the introduction of planet gearings (or simple differentials) in the conventional gearing. Practical and exact solutions can be obtained for arbitrary values of k and l, even for very large primes.

§ 1. Introduction

In special instruments or machines sometimes a gearing has to be realized of which the gear ratio is equal to the quotient of two integers k and l. If these integers are small, one or more useful solutions readily suggest themselves. If however this is not so, it is necessary to calculate suitable combinations of numbers of teeth. We will deal here only with this mathematical part of the design of gearing.

The problem how to find such numbers has been treated by two methods, which will be described briefly in the next §§. These are:

1. decomposition of k and l into prime factors.
2. approximation to the fraction k/l by continued fractions.

§ 2. Decomposition into prime factors

Trying to factorize a number of the order of 10.000 or more is a rather laborious task. Time and labour can be saved by the use of a suitable table [1—5].

The decomposition being completed, we are left with prime factors $k_1, k_2 \ldots k_p$ of k and $l_1, l_2 \ldots . l_q$ of l. If all these factors are sufficiently small as compared with the maximal number N of teeth allowed on a wheel, several combinations of these factors over each pair of wheels can be tried. If for instance $p = 4$ and $q = 5$, we may take wheels with numbers of teeth equal to $k_1 \times k_2, k_3, k_4, l_1 \times l_3, l_2 \times l_4, l_5$, combining the wheels as

$$\frac{k_1 \times k_2}{l_1 \times l_3}, \frac{k_3}{l_2 \times l_4}, \frac{k_4}{l_5}, \text{ or as } \frac{k_1 \times k_2}{l_2 \times l_4}, \frac{k_3}{l_5}, \frac{k_4}{l_1 \times l_3}, \text{ etc.}$$

An arbitrary suitable factor may be introduced both in the numerator and in the denominator of each of these fractions. We may try to obtain a minimal number of wheels and standard numbers of teeth.

If some of the factors are not small as compared with N, we are forced to use wheels with numbers of teeth equal to these factors. As these are prime numbers they will often not be found among standard tooth numbers and it will be rather difficult to obtain or make such wheels.

If at least one of the factors is larger than N, this type of solution becomes impossible. This is also the case if one or both decompositions cannot be found. For integers of about 10^5 or more tables will not generally be at hand. For numbers in excess of 10^8 the decomposition is even unknown.

A solution found with this method is always an exact one, in other words the error in the gear ratio is zero.

§ 3. Approximation by continued fractions

By the arithmetical method of developing the fraction k/l in continued fractions [6, 7] we obtain a finite series of fractions k'/l' of integers which give successive approximations to k/l. We have always $k' < k$ and $l' < l$.

If the method of § 2 gives no solution for k/l, this method can be tried now for each of the fractions k'/l'. The problem is at least somewhat simplified, as we have smaller numbers [8].

The surprising feature in this method is that often sufficient or even very good approximations can be found with far smaller numbers. See the example in § 7.

There are however cases in which only an exact solution can be used. If the rotation is always in the same direction even a very small error in the ratio will give rise to an ever increasing difference between the actual position of the output shaft and its desired position. Also it may happen that all fractions which give sufficient approximations cannot be used for reasons mentioned in § 2.

§ 4. Consideration of conventional gearing

In a conventional gearing we have an input shaft which is coupled by means of two meshed gears with the first intermediate shaft and so on to the output shaft. So there is one single path between in- and output. Each of the pairs of gears multiplies or divides the velocity by a fraction which is the ratio of the numbers of teeth on the wheels. Looking upon this apparatus as a calculating machine we could say that it can multiply or divide only. It is exactly this property that forces us to the decomposition of § 2 if we wish to realize a given gear ratio. Of course one uses this scheme because multiplication can be done mechanically in such a simple way.

There is however also a rather simple mechanical device which can add and subtract. This is the planet gearing, or in its simplest form the differential. This line of thought indeed leads to results.

By making use of differentials in gearings as considered here many degrees of freedom in the design are obtained. So many indeed, that it is even (at least theoretically) possible to realize a gearing for any given gear ratio k/l in which all wheels have the same (arbitrary) number of teeth! This will be shown in § 6.

If a gearing with a given gear ratio has to be designed we suggest starting in the usual way, that is as described in § 2. If an exact solution is not necessary the method of § 3

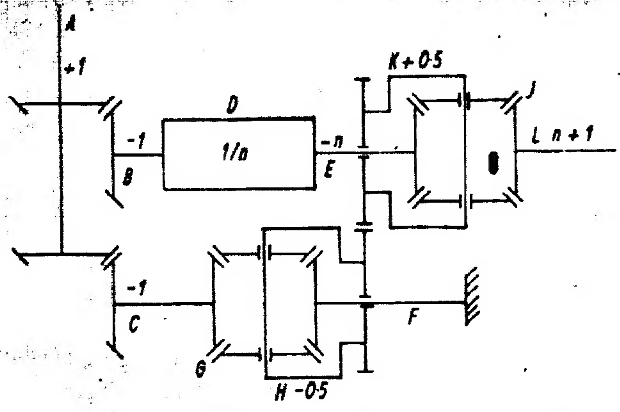

Fig. 1. A gearing D with a gear ratio of $1/n$ can be modified to a gearing as shown, with a gear ratio of $1/(n+1)$, by the use of wheels with the same (and arbitrary) number of teeth only. G and J are differentials with planet pinions in cages H and K respectively. The shaft F is fixed. A and L are in- and output shafts.

should be applied next. As differentials are a complication, they should not be introduced before conventional design has been tried thoroughly.

The introduction of differentials turns out to be so powerful a means that in all cases several relatively simple and exact solutions can be found, independently of the possibility of decomposition or approximation [9].

In a gearing containing differentials the input shaft is coupled to more than one of the intermediate shafts. So the path between in- and output is not a single one. The scheme of such a gearing may be compared with an electronic amplifier containing positive or negative feed back loops.

§ 5. Calculation of a gearing with differentials for a given gear ratio k/l

We first decompose k and l into prime factors as far as possible. All prime factors larger than N may be written as an algebraical sum of two or more integers. These should be chosen in such a way that they can be decomposed themselves into some of the prime factors already present. The number of + and — signs in the terms should be kept as low as possible, since the number of differentials needed is equal to or larger than it. There are an infinite number of ways of writing arbitrary integers as a sum of even two suitably decomposable integers. A general rule which of these is the best one cannot (yet?) be given.

This being done, the scheme of the gearing generally suggests itself from the numerical form obtained. In § 7 we will give an example.

§ 6. Arbitrary gear ratio k/l obtained by wheels with equal numbers of teeth

The proof that this is actually possible consists of three parts.

First we argue that if we have to realize a ratio k/l we are able to do so, as soon as we can realize $1/n$ in which n is an arbitrary number. For if we have $1/n$ we have also $1/n$ by interchanging in- and output shafts). As n is arbitrary we can realize $k/1$ (case $n = k$) and $1/l$ (case $n = l$). By coupling the output shaft of the gearing with ratio $k/1$ to the input shaft of that with ratio $1/l$ the ratio of the series gearing becomes k/l. So we have only to prove that a ratio $1/n$ is possible.

The second step is to point out that we can realize a gear ratio $1/n$ for arbitrary n if we can prove two things:

1. 1/1 can be realized,
2. a ratio $1/n$ can be modified into $1/(n+1)$.

This is what is generally called an inductive proof. Once these two facts are established we can modify 1/1 into 1/2, this into 1/3 and so on.

Now obviously a ratio 1/1 exists. This is a single shaft.

So the third and final step will be to prove that a gear ratio $1/n$ can be changed into $1/(n+1)$ for arbitrary values of n by the use of wheels with the same number of teeth only. It turns out that two differentials have to be introduced to provide for this change.

Fig. 1 is a schematical drawing for this case. If the input shaft A makes + 1 revolution, B and C make —1 revolution. D is the gearing with the ratio $1/n$. So E will make $-n$ revolutions. F is fixed. So the cage H with the two planet pinions of the differential G will make —0.5 revolution. This will cause cage K with the two planet pinions of differential J to make + 0.5 revolution. The output shaft L will turn, due to the rotation of E and K over $-(-n) + 2 \times 0.5 = n + 1$ revolutions. All wheels may have the same number of teeth. Now our proof is complete.

Of the two differentials G and J only J serves for addition. Its function corresponds with the + sign in $1/(n+1)$. However in the output shaft L we find the rotation of the cage K multiplied by a factor two and so we need the other differential G to correct for this effect. It serves merely as a gearing with a gear ratio 2/1.

There are two paths A-B-D-E-J-L and A-C-G-H-K-J-L from the input shaft A to the output shaft L. If n is large the former can be considered as the main path, and the other as a correction to it (positive feed „forward" coupling). The whole system has mechanically one degree of freedom only. Therefore we may as well consider L as the input shaft. The gear ratio is $(n+1)/1$ in that case, and the path C-G-H-K-J can be considered as a feedback path. As the number of revolutions of the input shaft L is reduced by it from $n+1$ to n, the feed back is negative here.

The reader may try to draw a scheme on this basis for a gear ratio of say 5/7.

If in this general gearing for a gear ratio of k/l, k and l are not very small, a large number of the intermediate shafts will rotate at a very low speed as compared with the in- and output shafts. This leads to difficulties in the next (mechanical) stage of the design. So this general gearing has very little practical value. It only serves to demonstrate to what extent we have liberated ourselves of the conditions imposed on the numbers of teeth in the conventional methods.

§ 7. Example for comparing the methods described

Let us suppose as an arbitrary example that a gearing has to be realized with a gear ratio of 1631/391.

The method of decomposition into prime factors gives $1631 = 7 \times 233$ and $391 = 17 \times 23$. A number 233 of teeth on a wheel will not be tolerable.

The method of approximation by continued fractions gives the fractions listed below. For each of these fractions we have given the decomposition of the numerator and the denominator into prime factors, the numerical value, and the relative error of the approximation. For comparison such values have also been given for 1631/391 in the last line (the only exact solution obtainable with the conventional method).

Fraction	Decomposition	Numerical Value	Relative Error
5/1	5/1	5.000,000	$+2 \times 10^{-1}$
4/1	$2 \times 2/1$	4.000,000	-4×10^{-2}
21/5	$3 \times 7/5$	4.200,000	$+7 \times 10^{-3}$
25/6	$5 \times 5/2 \times 3$	4.166,667	-1×10^{-3}
171/41	$3 \times 3 \times 19/41$	4.170,732	-2×10^{-4}
146/35	$2 \times 73/5 \times 7$	4.171,429	$+2 \times 10^{-5}$
1631/391	$7 \times 233/17 \times 23$	4.171,356	0

From this list it is clear that if better approximations are needed larger prime factors will occur in the numbers of teeth. Perhaps 146/35 can be accepted in this respect, if an exact solution is not necessary.

Next we will demonstrate that the occurrence of the factor 233 can be avoided by introduction of one differential, the gear ratio remaining exact. We write 1631 as an algebraical

sum of two integers, one having 17 and the other having 23 as a prime factor. Suitable solutions turn out to be:

$1631/391 = (1700-69)/391 = 1700/391-69/391 = 100/23-$
$-3/17$,

$1631/391 = (1495 + 136)/391 = 1495/391 + 136/391 =$
$= 65/17 + 8/23$.

Fig. 2 is a schematical drawing of a realization of this gearing based on the second solution. A is the input shaft, L the output shaft. Wheels D, E, K, J, H and G have numbers of teeth respectively equal to 23, 16, 65, 17, 16 and 64. These numbers have been shown in brackets in the figure. The action of this gearing can best be understood starting from L. Suppose L turns over $+ 391$ revolutions. J will make $-391 \times 65/17 = -1495$ revolutions. F will make $-391 \times 16/64 = -391/4$ revolutions. The cage C with the two planet pinions of the differential B will make $(391/4) \times 16/23 = + 68$ revolutions. Due to the rotations of J and C, A will turn over $+ 1495 + + 2 \times 68 = 1631$ revolutions. So if A is the input shaft the gear ratio will be 1631/391 exactly.

This gearing can be considered as an extreme simplification of the gearing of § 6 for this case. The latter one contains for our example $2 \times (1630 + 390) = 4040$ differentials!

It is advantageous to use a solution in which there is a main path with as few wheels as possible, providing a gear ratio almost equal to the desired one. This path (A-B-J-K-L in our example) transmits a substantial part of the flow of energy from the input- to the output shaft with a minimal amount of back lash and elastical deformation. The other paths merely act to give a correction (less than 10% in our example) in the gear ratio. So at least some of the wheels in the latter paths may be much smaller or lighter. In this respect the first of the two solutions shown above is better than the one used in our example. Moreover it only needs to have two pairs of meshed gears besides the differential (with numbers of teeth for instance equal to 12, 68, 23 and 50) against the three in our example. The gear ratio will be negative however, that is the input and the output shaft rotate in opposite directions.

§ 8. Applications

A gearing with two differentials for a gear ratio of 1.831,211/1.826,211 has been calculated by the author. This gearing served as a coupling between the second-hands of two dials of a clock, one indicating sidereal time, the other mean solar time. The gear ratio had to be equal to the best known value of the ratio of the lengths of the mean solar day and the sidereal day, given by the seven-place numbers mentioned above. One of the hands had to be coupled to a clock of good quality, adjusted every day. The indication on the other dial, once adjusted, will automatically remain correct for its type of time with an error of less than 1 second/year.

The large numbers mentioned could not be decomposed into sufficiently small prime factors. In the calculation of this gearing no such decomposition has been necessary. Besides the two differentials 18 wheels were needed. The largest number of teeth was 109. Of course even this number could have been eliminated by the introduction of a third differential. The main path consisted of a coupling 1/1 through a differential. This gave the necessary correction of about 0.3%. All 18 wheels were in the correction path.

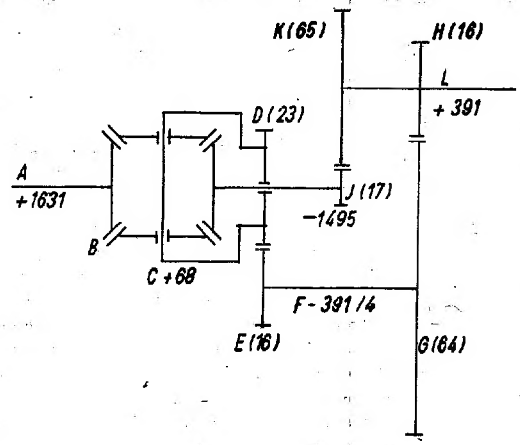

Fig. 2. Gearing with a gear ratio of exactly $1631/391 = 7 \times 233/17 \times 23$. B is a differential with planet pinions in cage C. Numbers of teeth have been indicated in brackets. Note the absence of a wheel with 233 teeth.

A schematical drawing of this gearing was presented in July 1950 to the board of the Dr A. F. Philips' Sterrewacht (Astronomical Observatory) at Eindhoven.

The idea has been applied independently in U.S.A. for the calibration of precision screws and control of ruling engines [10].

It is interesting to note that application of this method to the calculation of special gearing could lead to a certain standardization of toothed wheels or wheel combinations new for such gearing. If we should have available the gear ratios 1/2, 1/3 and 1/4 and moreover 1/10, 1/100, 1/1000 etc., we could compose a gearing with every desired ratio by a suitable combination of multiplication and addition units.

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